ON THE QUADRATIC INTEGRAL OF THE EQUATIONS OF MOTION OF A BODY WITH A FIXED POINT

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The conditions for the existence of a fourth algebraic integral of this problem were studied in the works of S.V. Kovalevskaia, A.M. Liapunov, G.G. Appelrot, A. Poincaré, E. Husson, P. Burgatti, P.Ia. Kochina (see, for example [1 and 2]). These studies assume that the fourth integral does not depend explicitely on time and that in common with the other three integrals it contains an arbitrary constant.

Although solutions having algebraic invariant relations(*) are discovered from time to time(**), the conditions for their existence have not been established even for the simplest cases. Chaplygin examined the conditions for the existence of solutions with linear invariant relations [5]. The results of the latter paper were made more precise by Kharlamov [6]. The first solution with a quadratic invariant relation was found by Steklov [7] and then followed those of Goriachev [8], Chaplygin [9] and Kowalewski [10]. Kharlamov working from his own equations [11, 12] studied the conditions for the existence of solutions with two invariants, one of them quadratic [13]. Later he noted that a second invariant relation in the general case must have the form of a rational function, and that is, it must be the ratio of a fourth degree polynomial to a second degree polynomial [14]; however in the papers [13 and 14] the second invariant relation was taken as a polynomial.

In the present paper the latter restriction is removed.

Let us write the equations presented in [13] using the same notation:

$$\begin{gathered}[2Q+(A-B)p^2+2\lambda p]\frac{dR}{dp}-[2R+(A-C)p^2+2\lambda p]\frac{dQ}{dp}+ \\ +(Ap+\lambda)\left[Q-R+\frac{(C-B)}{A}E\right]=\frac{(B-C)}{A}k \\ [2Q+(A-B)p^2+2\lambda p]\frac{d^2R}{dp^2}+\left[\frac{dQ}{dp}+(A-B)p+\lambda\right]\frac{dR}{dp}+ \\ +Cp\frac{dQ}{dp}-\frac{Q-R}{B-C}AC+CE=0\end{gathered}\qquad(1)$$

$$q^2\frac{(C-B)}{A}B=(A-C)p^2+2\lambda p+2R\qquad(2)$$

From the last expression we note that R is a quadratic function of p and q and we will stipulate a quadratic invariant relation of the form

$$R=c_2p^2+c_1p+c_0\qquad(3)$$

Substituting R, dR/dp, d^2R/dp^2 into (1) we find that Q and dQ/dp are rational functions of p of the form:

$$Q=P_4\,/\,P_2,\quad dQ\,/\,dp=P_3\,/\,P_2$$

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*) This term was used in [3].

**) The most recent solution is indicated by Kharlamova in [4] where all the integrable solutions known up to now are presented.

where P_k is a polynomial in p of degree k. Hence for $dP_2/dp \not\equiv 0$ we get:

$$Q = \frac{P_4}{P_2} \equiv \frac{dP_4/dp - P_3}{dP_2/dp} = \frac{a_3p^3 + a_2p^2 + a_1p + a_0}{\gamma p + \delta} \tag{4}$$

where a_i, γ, δ are known functions of c_i and of the system parameters.

The cases $dP_2/dp \equiv 0$ and $\gamma = 0$ will not be examined since they reduce to already known solutions ($dP_2/dp \equiv 0$ was solved by Goriachev and $\gamma = 0$ is the second solution given in [13]). Then the invariant relationship (4) may be written in the form

$$Q = b_2p^2 + b_1p + b_0 + \beta / (p + \alpha) \tag{5}$$

In the following we will assume $\beta \neq 0$ since $\beta = 0$ has already been solved in [13] for $n = 2$.

Expressions (3) and (5) must transform (1) into identities leading to the following conditions:

$$\begin{gathered}
(A - 2B)c_2 - (A - 2C)\, b_2 = 0 \\
8\, c_2b_2 + 2Cb_2 + 4(A - B)c_2 - (b_2 - c_2)AC/(B - C) = 0 \\
2c_2b_1 - 2c_1b_2 - Bc_1 + Cb_1 + 3\lambda\, (c_2 - b_2) = 0 \\
6\, (b_1 + \lambda)c_2 + (A - B)c_1 + 2c_1b_2 + Cb_1 - (b_1 - c_1)AC/(B - C) = 0 \\
4c_2b_0 - 4c_0b_2 + A\, (b_0 - c_0) + \lambda\, (c_1 - b_1) + (C - B)\, E = 0 \\
4c_2b_0 + c_1\, (b_1 + \lambda) - (b_0 - c_0)AC/(B - C) + CE = 0
\end{gathered} \tag{6}$$

$$\begin{gathered}
2c_0\beta + \alpha\beta\, (2c_1 + \lambda) + \alpha^2 N = 0 \\
2c_2 - C - AC/(B - C) = 0, \qquad c_1 - \alpha\, [4c_2 - AC/(B - C)] = 0 \\
\beta\, (4c_1 + 3\lambda) + \alpha\, [2N + \beta\, (4c_2 + A)] = 0
\end{gathered} \tag{7}$$

In these

$$N = 2c_1b_0 - 2c_0b_1 + \lambda\,(b_0 - c_0) + \frac{\lambda\,(C - B)}{A}\, E - \frac{(B - C)}{A}\, k$$

The quantities b_2, b_1, b_0, c_2, c_1, c_0 are found from (6) and have the values given in Section 3 of [13].

From the conditions (7) we find

$$B = \frac{A^2 - AC + C^2}{A + C}, \qquad H = -\frac{\lambda^2}{(A - C)(2C - A)^4}\,(A^4 - A^3C - 4A^2C^2 + 9AC^3 - 4C^4) \tag{8}$$

The constant H was introduced instead of E:

$$H = \frac{(b_0 - c_0)}{B - C}\, A - E$$

Under the conditions (8) the values of c_2, c_1, c_0 are:

$$c_2 = \frac{C\,(2A - C)}{2\,(A - 2C)}, \qquad c_1 = \frac{3C\,(A - C)}{(A - 2C)^2}\,\lambda, \qquad c_0 = \frac{A^2 - AC + C^2}{2\,(A - 2C)^3}\,\lambda^2$$

and the relation (2) takes the form

$$q^2 + \frac{(A + C)^2}{(A - 2C)^2}\left(p + \frac{\lambda}{A - 2C}\right)^2 = 0$$

In an actual motion this is possible only with constant values of p and q.

Consequently, Eqs. (1) allow only three solutions [13] with the quadratic invariant relation (3).

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